

NORMALITY AND SHARING VALUES

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ABSTRACT. In this paper, we obtain some normality criteria for families of holomorphic functions. these generalize some results of Fang, Xu, Chen and Hua.

1. INTRODUCTION

We denote the complex plane by \mathbb{C} , and the unit disk by Δ . Let f be a meromorphic function in \mathbb{C} . We say that f is a normal function if there exists a positive M such that $f^\#(z) \leq M$ for all $z \in \mathbb{C}$, where $f^\# = \frac{|f'(z)|}{1+|f(z)|^2}$ denotes the spherical derivative of f .

A family \mathcal{F} of analytic functions on a domain $\Omega \subseteq \mathbb{C}$ is normal in Ω if every sequence of functions $f_n \in \mathcal{F}$ contains either a subsequence which converges to a limit function $f \neq \infty$ uniformly on each compact subset of Ω , or a subsequence which converges uniformly to ∞ on each compact subset.

In this paper, we use the following standard notation of value distribution theory,

$$T(r, f); m(r, f); N(r, f); \overline{N}(r, f), \dots$$

We denote $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\}, \text{ as } r \rightarrow +\infty,$$

possibly outside of a set with finite measure.

According to Bloch's principle every condition which reduces a meromorphic function in the plane to a constant, makes the family of meromorphic functions in a domain G normal. Rubel gave four counter examples to Bloch principle.

Let f and g be two meromorphic functions in a domain D and $a \in \mathbb{C}$. If $f - a$ and $g - a$ have the same number of zeros in D (ignoring multiplicity). Then we say that f and g share the value $z = a$ IM.

Let us recall the following known results that establish connection between shared values and normality.

Mues and Steinmetz proved the following result.

Theorem 1.1. [9] *Let f be a non constant meromorphic function in the plane. If f and f' share three distinct complex numbers a_1, a_2, a_3 then $f \equiv f'$.*

Wilhelm Schwick seems to have been the first to draw a connection between normality and shared values. He proved the following theorem

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Theorem 1.2. [8] *Let \mathcal{F} be a family of meromorphic functions on a domain G and a_1, a_2, a_3 be distinct complex numbers. If f and f' share a_1, a_2, a_3 for every $f \in \mathcal{F}$, then \mathcal{F} is normal in G .*

Chen and Hua proved the following theorem

Theorem 1.3. [6] *Let \mathcal{F} be a family of holomorphic functions in a domain D . Suppose that there exists a non zero $a \in \mathbb{C}$ such that for each function $f \in \mathcal{F}$; f, f' and f'' share the value $z = a$ IM in D . Then the family \mathcal{F} is normal in D .*

Fang and Xu improved their results by proving the following theorems

Theorem 1.4. [7] *Let \mathcal{F} be a family of holomorphic functions on a domain D and let a, b be two distinct finite complex numbers such that $b \neq 0$. If for any $f \in \mathcal{F}$, f and f' share $z = a$ IM and $f(z) = b$ whenever $f'(z) = b$ then \mathcal{F} is normal in D .*

Theorem 1.5. [7] *Let \mathcal{F} be a family of holomorphic functions in a domain D , and let a be a non zero finite complex number. If for any $f \in \mathcal{F}$ f and f' share $z = a$ IM and $f^{(k)}(z) = a, f^{(k+1)}(z) = a$ whenever $f(z) = a$. Then \mathcal{F} is normal in D .*

Finally, Fang proved the following.

Theorem 1.6. [10] *Let \mathcal{F} be a family of meromorphic functions in a domain D and let $a(z)$ be a non vanishing analytic function in D . If, for every function $f \in \mathcal{F}$, f and f' have the same zeros, and $f(z) = a(z)$ whenever $f'(z) = a(z)$, then \mathcal{F} is normal in D .*

More recently Xia and Xu improved theorem 1.6 by showing the following :

Theorem 1.7. [12] *Let \mathcal{F} be a family of meromorphic functions in a domain D , and k be a positive integer, and let $\varphi(z) (\neq 0, \infty)$ be a non vanishing meromorphic function in D such that f and $\varphi(z)$ have no common zeros for all $f \in \mathcal{F}$ and $\varphi(z)$ has no simple zeros in D , and all poles of $\varphi(z)$ have multiplicity at most k . If, for each $f \in \mathcal{F}$,*

(1) *all zeros of f have multiplicity at least $k + 1$*

(2) *$f(z) = 0$ whenever $f^{(k)}(z) = 0$ and $f(z) = \varphi(z)$ whenever $f^{(k)}(z) = \varphi(z)$*

then \mathcal{F} is normal in D .

2. MAIN THEOREMS AND LEMMAS

We improve Theorem 1.4 and Theorem 1.5 by showing the following.

Theorem 2.1. *Let \mathcal{F} be a family of holomorphic functions on a domain D such that all zeros of $f \in \mathcal{F}$ are of multiplicity at least k , where k is a positive integer. Let a, b be two distinct finite complex numbers such that $b \neq 0$. Suppose for any $f \in \mathcal{F}$ satisfies the following*

(1): *f and $f^{(k)}$ share $z = a$ IM*

(2): *$f(z) = b$ whenever $f^{(k)}(z) = b$*

then \mathcal{F} is normal in D .

One may ask whether we can replace the values a and b by holomorphic functions. We show in the following theorem that this is indeed the case.

Theorem 2.2. *Let \mathcal{F} be a family of holomorphic functions on a domain D such that all zeros of $f \in \mathcal{F}$ are of multiplicity at least k , where k is a positive integer. Let $a(z), b(z), \alpha_0(z), \alpha_1(z)$ be holomorphic functions in D , with $\alpha_0(z) \neq 0$. If, for each $f \in \mathcal{F}$,*

(1): *$b(z) \neq 0$*

- (2): $a(z) \neq b(z)$, and $b(z) - \alpha_1(z)a(z) - \alpha_0(z)a^{(k)}(z) \neq 0$.
 (3): $f(z) = a(z)$ if and only if $\alpha_0(z)f^{(k)}(z) + \alpha_1(z)f(z) = a(z)$
 (4): $f(z) = b(z)$ whenever $\alpha_0(z)f^{(k)}(z) + \alpha_1(z)f(z) = b(z)$
 then \mathcal{F} is normal in D .

Remark 1: The hypothesis $a(z) \neq b(z)$ and $b(z) - \alpha_1(z)a(z) - \alpha_0(z)a^{(k)}(z) \neq 0$ can not be dropped in Theorem 2.2.

Example 1: Let $D = \Delta = \{z : |z| < 1\}$ and $a(z) = b(z) = z^{k-1}$, $\alpha_0(z) = 1$, $\alpha_1(z) = 0$ and

$$\mathcal{F} = \{e^{nz} - \frac{z^{k-1}}{n^k} + z^{k-1} : n = 1, 2, \dots\}.$$

Then for any $f \in \mathcal{F}$, and

$$f = e^{nz} - \frac{z^{k-1}}{n^k} + z^{k-1}, \quad f^{(k)} = n^k e^{nz}$$

Clearly, conditions of Theorem 2.2 are satisfied. However, \mathcal{F} is not normal in Δ .

This example confirms that $b(z) \neq 0$ is necessary in Theorem 2.2 as $f^{(k)}(z) \neq 0$.

Example 2: Let $D = \Delta = \{z : |z| < 1\}$, k be a positive integer, $b(z) = b$ (a non zero constant) and $a(z) = ((-1)^{k+1} + 1)b$ and

$$\mathcal{F} = \{b \frac{(z - \frac{1}{n})^k}{k!} + \frac{(-1)^{k+1}}{k!(z - \frac{1}{n})} + a : n = 1, 2, \dots\}$$

Then, for every $f_n(z) \in \mathcal{F}$,

$$f_n(z) = b \frac{(z - \frac{1}{n})^k}{k!} + \frac{(-1)^{k+1}}{k!(z - \frac{1}{n})} + a, \quad f_n^{(k)}(z) = b - \frac{1}{(z - \frac{1}{n})^{k+1}}$$

Clearly, f_n and $f_n^{(k)}$ share a and $f_n^{(k)}(z) \neq b$, so that $f_n(z) = b$ whenever $f_n^{(k)}(z) = b$. But \mathcal{F} is not normal in D .

Theorem 2.3. Let \mathcal{F} be a family of holomorphic functions in a domain D such that all zeros of $f \in \mathcal{F}$ are of multiplicity at least k , where k is a positive integer and let a be a non zero finite complex number. If for any $f \in \mathcal{F}$ f and $f^{(k)}$ share $z = a$ IM and $f^{(k+1)}(z) = a$ whenever $f(z) = a$. Then \mathcal{F} is normal in D .

We will use the tools of Fang and Xu which they used in their paper. For this we need the following.

Lemma 2.4. [4] [5] (Zalcman's lemma)

Let \mathcal{F} be a family of holomorphic functions in the unit disk Δ with the property that for every function $f \in \mathcal{F}$, the zeros of f are of multiplicity at least k . If \mathcal{F} is not normal at z_0 in Δ , then for $0 \leq \alpha < k$, there exist

- (a) a sequence of complex numbers $z_n \rightarrow z_0$, $|z_n| < r < 1$
- (b) a sequence of functions $f_n \in \mathcal{F}$ and
- (c) a sequence of positive numbers $\rho_n \rightarrow 0$

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges to a non constant entire function g on \mathbb{C} . Moreover g is of order at most one. If \mathcal{F} possesses the additional property that there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ for any $f \in \mathcal{F}$, then we can take $\alpha = k$.

Lemma 2.5. [1] [3] *Let f be a non constant meromorphic function. Then for $k \geq 1$, $b \neq 0, \infty$,*

$$T(r, f) \leq \overline{N}(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)} - b}) - N(r, \frac{1}{f^{(k+1)}}) + S(r, f)$$

3. PROOF OF THEOREM 2.1

Proof. Since normality is a local property, we assume that $D = \Delta = \{z : |z| < 1\}$ Suppose, \mathcal{F} is not normal in D ; without loss of generality we assume that \mathcal{F} is not normal at the point z_0 in Δ . Then by Lemma 2.4, there exist

(a) a sequence of complex numbers $z_n \rightarrow z_0$, $|z_n| < r < 1$

(b) a sequence of functions $f_n \in \mathcal{F}$ and

(c) a sequence of positive numbers $\rho_n \rightarrow 0$

such that $g_n(\zeta) = \rho_n^{-k} [f_n(z_n + \rho_n \zeta) - a]$

converges locally uniformly to a non constant entire function g . Moreover g is of order at most one.

Now we claim that $g = 0$ if and only if $g^{(k)} = a$ and $g^{(k)} \neq b$
Suppose, $g(\zeta_0) = 0$. then by Hurwitz's theorem there exist $\zeta_n; \zeta_n \rightarrow \zeta_0$ such that

$$g_n(\zeta_n) = \rho_n^{-k} [f_n(z_n + \rho_n \zeta_n) - a] = 0$$

Thus $f_n(z_n + \rho_n \zeta_n) = a$. Since f_n and $f_n^{(k)}$ share $z = a$ IM, we have

$$g_n^k(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a$$

Hence

$$g^{(k)}(\zeta_0) = \lim_{n \rightarrow \infty} g_n^{(k)}(\zeta_n) = a$$

Thus we have proved that $g^{(k)} = a$ whenever $g = 0$.

On the other hand, if $g^{(k)}(\zeta_0) = a$, then there exist $\zeta_n; \zeta_n \rightarrow \zeta_0$ such that
 $g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a$; $n = 1, 2, \dots$ hence $f_n(z_n + \rho_n \zeta_n) = a$ and $g_n(\zeta_n) = 0$ for $n=1, 2, \dots$ thus

$$g(\zeta_0) = \lim_{n \rightarrow \infty} g_n(\zeta_n) = 0$$

this shows that $g = 0$ whenever $g^{(k)} = a$

Hence $g = 0$ if and only if $g^{(k)} = a$.

Next, we prove $g^{(k)}(\zeta) \neq b$. Suppose that there exist ζ_0 satisfying $g^{(k)}(\zeta_0) = b$. Then, by Hurwitz's theorem, there exist a sequence $\zeta_n \rightarrow \zeta_0$ and $g_n^{(k)}(\zeta_n) = b$; $n = 1, 2, \dots$

Since $f_n(z) = b$ whenever $f_n^{(k)}(z) = b \Rightarrow f_n(z_n + \rho_n \zeta_n) = b$ and,

$$g_n(\zeta_n) = \rho_n^{-k} [f_n(z_n + \rho_n \zeta_n) - a] = \rho_n^{-k} [b - a] \rightarrow \infty, \text{ this contradicts}$$

$$\lim_{n \rightarrow \infty} g_n(\zeta_n) = g(\zeta_0) \neq \infty$$

So $g^{(k)}(\zeta) \neq b$. Hence we get,

$$(3.1) \quad g^{(k)}(\zeta) = b + e^{A\zeta+B}$$

where A and B are two constants. We claim that $A = 0$. Suppose that $A \neq 0$; then

$$(3.2) \quad g(\zeta) = \frac{b\zeta^k}{k!} + \frac{e^{A\zeta+B}}{A^k} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \dots + c_{k-1}\zeta + c_k$$

where c_1, c_2, \dots, c_k are constants. Let $g^{(k)} = a$. Then by (3.1), (3.2) and $g(\zeta) = 0$ whenever $g^{(k)}(\zeta) = a$, we have

$$\frac{b\zeta^k}{k!} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \dots + c_k + \frac{b-a}{A^k} = 0$$

This is a polynomial of degree k in ζ this polynomial has k solutions which contradicts the fact that $g^{(k)}$ has infinitely many solutions. Thus we have,

$$g^{(k)}(\zeta) = b + e^B$$

And

$$g(\zeta) = (b + e^B) \frac{\zeta^k}{k!} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \dots + c_k$$

Since g is non constant, this contradicts $g(\zeta) = 0 \Leftrightarrow g^{(k)}(\zeta) = a$. Thus \mathcal{F} is normal in D . This completes the proof of theorem. \square

4. PROOF OF THEOREM 2.2

Proof. Suppose that \mathcal{F} is not normal at $z_0 \in \Delta$, then by Lemma 2.4, there exist

(a) a sequence of complex numbers $z_n \rightarrow z_0$, $|z_n| < r < 1$

(b) a sequence of functions $f_n \in \mathcal{F}$ and

(c) a sequence of positive numbers $\rho_n \rightarrow 0$

such that $g_n(\zeta) = \rho_n^{-k} [f_n(z_n + \rho_n\zeta) - a(z_n + \rho_n\zeta)]$

converges locally uniformly to a non constant entire function g . Moreover g is of order at most one.

Now we claim that

(a): $g(\zeta) = 0 \Leftrightarrow g^{(k)}(\zeta) = \varphi(z_0)$, where $\varphi(z) = \frac{a(z) - \alpha_1(z)a(z) - \alpha_0(z)a^{(k)}(z)}{\alpha_0(z)}$

(b): $g^{(k)}(\zeta) \neq B$, where $B = \frac{b(z_0) - \alpha_1(z_0)a(z_0) - \alpha_0(z_0)a^{(k)}(z_0)}{\alpha_0(z_0)}$ Note that B is a constant.

Since

$$(4.1) \quad g_n(\zeta) = \rho_n^{-k} [f_n(z_n + \rho_n\zeta) - a(z_n + \rho_n\zeta)] \rightarrow g(\zeta)$$

we have

$$(4.2) \quad g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) - a^{(k)}(z_n + \rho_n\zeta) \rightarrow g^{(k)}(\zeta)$$

Now suppose that $g(\zeta_0) = 0$. Then by Hurwitz's theorem, there exists ζ_n , $\zeta_n \rightarrow \zeta_0$ such that

$$g_n(\zeta_n) = \rho_n^{-k} [f_n(z_n + \rho_n\zeta) - a(z_n + \rho_n\zeta)] = 0.$$

Thus

$$f_n(z_n + \rho_n\zeta) = a(z_n + \rho_n\zeta).$$

Since $f(z) = a(z) \Leftrightarrow \alpha_0(z)f^{(k)}(z) + \alpha_1(z)f(z) = a(z)$, we have $\alpha_0(z)f^{(k)}(z) + \alpha_1(z)f(z) = a(z)$.

Also

$$\begin{aligned}
 \frac{\alpha_0(z_n + \rho_n \zeta)f^{(k)}(z_n + \rho_n \zeta) + \alpha_1(z_n + \rho_n \zeta)f(z_n + \rho_n \zeta)}{\alpha_0(z_n + \rho_n \zeta)} &= f_n^{(k)}(z_n + \rho_n \zeta) + \frac{\alpha_1(z_n + \rho_n \zeta)}{\alpha_0(z_n + \rho_n \zeta)} f_n(z_n + \rho_n \zeta) \\
 &= f_n^{(k)}(z_n + \rho_n \zeta) + \frac{\alpha_1(z_n + \rho_n \zeta)}{\alpha_0(z_n + \rho_n \zeta)} [\rho_n g_n(\zeta) + a(z_n + \rho_n \zeta)] \\
 (4.3) \quad &\rightarrow g^{(k)}(\zeta) + a^{(k)}(z_0) + \frac{\alpha_1(z_0)}{\alpha_0(z_0)} a(z_0)
 \end{aligned}$$

Therefore it follows that,

$$\begin{aligned}
 g^{(k)}(\zeta_0) &= \lim_{n \rightarrow \infty} \left[\frac{\alpha_0(z_n + \rho_n \zeta)f^{(k)}(z_n + \rho_n \zeta) + \alpha_1(z_n + \rho_n \zeta)f(z_n + \rho_n \zeta)}{\alpha_0(z_n + \rho_n \zeta)} \right] - a^{(k)}(z_0) - \frac{\alpha_1(z_0)}{\alpha_0(z_0)} a(z_0) \\
 &= \lim_{n \rightarrow \infty} \frac{a(z_n + \rho_n \zeta)}{a_0(z_n + \rho_n \zeta)} - a^{(k)}(z_0) - \frac{\alpha_1(z_0)}{\alpha_0(z_0)} a(z_0) \\
 &= \frac{a(z_0) - \alpha_1(z_0)a(z_0) - \alpha_0(z_0)a^{(k)}(z_0)}{\alpha_0(z_0)} = \varphi(z_0).
 \end{aligned}$$

Hence we have proved $g^{(k)}(\zeta) = \varphi(z_0)$ whenever $g(\zeta) = 0$

On the other hand, if $g^{(k)}(\zeta_0) = \varphi(z_0)$ then there exists ζ_n ; $\zeta_n \rightarrow \zeta_0$, such that

$$f_n^{(k)}(z_n + \rho_n \zeta) - a^{(k)}(z_n + \rho_n \zeta) = \varphi(z_0)$$

We have to show

$$g(\zeta_0) = \lim_{n \rightarrow \infty} g_n(\zeta_n) = \lim_{n \rightarrow \infty} [f_n(z_n + \rho_n \zeta_n) - a(z_n + \rho_n \zeta_n)] = f(z_0) - a(z_0) = 0$$

Now, assume that $g^{(k)}(\zeta_0) = \varphi(z_0)$ by using assumption (3) of the Theorem we get $f(z_0) - a(z_0) = 0$, so is $g(\zeta_0) = 0$. This shows that $g(\zeta) = 0 \Leftrightarrow g^{(k)}(\zeta) = \varphi(z_0)$

From (4.3) we deduce that

$$\begin{aligned}
 \frac{\alpha_0(z_n + \rho_n \zeta)f^{(k)}(z_n + \rho_n \zeta) + \alpha_1(z_n + \rho_n \zeta)f(z_n + \rho_n \zeta) - b(z_n + \rho_n \zeta)}{\alpha_0(z_n + \rho_n \zeta)} &\rightarrow g^{(k)}(\zeta) + a^{(k)}(z_0) + \frac{\alpha_1(z_0)}{\alpha_0(z_0)} a(z_0) - \frac{b(z_0)}{\alpha_0(z_0)} \\
 (4.4) \quad &= g^{(k)}(\zeta) - \frac{b(z_0) - \alpha_0(z_0)a^{(k)}(z_0) - \alpha_1(z_0)a(z_0)}{\alpha_0(z_0)} = g^{(k)}(\zeta) - B
 \end{aligned}$$

Next we prove that $g^{(k)}(\zeta) \neq B$. Suppose that there exists ζ_0 satisfying $g^{(k)}(\zeta_0) = B$. Then, by Hurwitz's theorem, there exists a sequence ζ_n ; $\zeta_n \rightarrow \zeta_0$ and by (4.4)

$$\{\alpha_0(z_n + \rho_n \zeta_n)f^{(k)}(z_n + \rho_n \zeta_n) + \alpha_1(z_n + \rho_n \zeta_n)f(z_n + \rho_n \zeta_n)\} - b(z_n + \rho_n \zeta_n) = 0$$

From the assumption, we have $f_n(z_n + \rho_n \zeta_n) = b(z_n + \rho_n \zeta_n)$. Then we get

$$\begin{aligned} g(\zeta_0) &= \lim_{n \rightarrow \infty} \rho_n^{(k)} [f_n(z_n + \rho_n \zeta_n) - a(z_n + \rho_n \zeta_n)] \\ &= \lim_{n \rightarrow \infty} \rho_n^{(k)} [b(z_n + \rho_n \zeta_n) - a(z_n + \rho_n \zeta_n)] = \infty \end{aligned}$$

which is a contradiction. So $g^{(k)}(\zeta) \neq B$.

Hence we get

$$(4.5) \quad g^{(k)}(\zeta) = B + e^{A\zeta + D}$$

where A and D are two constants. We claim that $A = 0$. Suppose that $A \neq 0$; then

$$(4.6) \quad g(\zeta) = \frac{B\zeta^k}{k!} + \frac{e^{A\zeta + D}}{A^k} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \dots + c_{k-1}\zeta + c_k$$

where c_1, c_2, \dots, c_k are constants. Let $g^{(k)}(\zeta) = \varphi(z_0)$ then by (4.5), (4.6) and $g(\zeta) = 0$ if and only if $g^{(k)}(\zeta) = \varphi(z_0)$

So we get

$$\frac{B\zeta^k}{k!} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \dots + c_k + \frac{B - \varphi(z_0)}{A^k} = 0$$

This is a polynomial of degree k in ζ this polynomial has k solutions which contradicts the fact that $g^{(k)}$ has infinitely many solutions. Thus we have,

$$g^{(k)}(\zeta) = B + e^D$$

And

$$g(\zeta) = (B + e^D) \frac{\zeta^k}{k!} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \dots + c_k$$

Since g is non constant, this contradicts $g(\zeta) = 0$ if and only if $g^{(k)}(\zeta) = \varphi(z_0)$. Thus \mathcal{F} is normal in D . This completes the proof of theorem. \square

5. PROOF OF THEOREM 2.3

Proof. Suppose \mathcal{F} is not normal in Δ ; without loss of generality we assume that \mathcal{F} is not normal at the point $z = 0$. Then by Lemma 2.4, there exist

(a) a sequence of complex numbers $z_n \rightarrow 0$, $|z_n| < r < 1$

(b) a sequence of functions $f_n \in \mathcal{F}$ and

(c) a sequence of positive numbers $\rho_n \rightarrow 0$

such that $g_n(\zeta) = \rho_n^{-k} [f_n(z_n + \rho_n \zeta) - a]$

converges locally uniformly to a non constant entire function g . Moreover g is of order at most one.

Now we claim that $g = 0$ iff $g^{(k)} = a$ and $g^{(k+1)} = 0$ whenever $g = 0$

Let $g(\zeta_0) = 0$. Then by Hurwitz's theorem there exist $\zeta_n; \zeta_n \rightarrow \zeta_0$ such that

$$g_n(\zeta_n) = \rho_n^{-k} [f_n(z_n + \rho_n \zeta_n) - a] = 0$$

Thus $f_n(z_n + \rho_n \zeta_n) = a$ since f_n and $f_n^{(k)}$ share $z = a$ IM, we have

$$g_n^k(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a$$

and

$$g_n^{(k+1)}(\zeta_n) = \rho_n f_n^{(k+1)}(z_n + \rho_n \zeta_n)$$

which implies that

$$g^{(k)}(\zeta_0) = \lim_{n \rightarrow \infty} g_n^{(k)}(\zeta_n) = a$$

and

$$g^{(k+1)}(\zeta_0) = \lim_{n \rightarrow \infty} g_n^{(k+1)}(\zeta_n) = 0$$

Thus we get, $g^{(k)} = a$ whenever $g = 0$ and $g^{(k+1)} = 0$ whenever $g = 0$.

On the other hand, if $g^{(k)}(\zeta_0) = a$ then there exist $\zeta_n \rightarrow \zeta_0$ such that $g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a$, $n = 1, 2, \dots$ hence $f_n(z_n + \rho_n \zeta_n) = a$ and $g_n(\zeta_n) = 0$ for $n=1, 2, \dots$ thus

$$g(\zeta_0) = \lim_{n \rightarrow \infty} g_n(\zeta_n) = 0.$$

This shows that $g = 0$ whenever $g^{(k)} = a$.

Hence $g = 0$ if and only if $g^k = a$ and $g^{(k+1)} = 0$ whenever $g = 0$.

Now using Lemma 2.5 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} T(r, g) &\leq \overline{N}(r, g) + N(r, \frac{1}{g}) + N(r, \frac{1}{g^{(k)} - a}) - N(r, \frac{1}{g^{(k+1)}}) + S(r, g) \\ &= N(r, \frac{1}{g}) + N(r, \frac{1}{g^{(k)} - a}) - N(r, \frac{1}{g^{(k+1)}}) + S(r, g) \\ &\leq N(r, \frac{1}{g^{(k)} - a}) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g) \\ &\leq T(r, \frac{1}{g^{(k)} - a}) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g) \\ &\leq T(r, g^{(k)} - a) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g) \end{aligned}$$

$$(5.1) \quad \leq T(r, g) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g)$$

Thus we get

$$(5.2) \quad \overline{N}\left(r, \frac{1}{g^{(k+1)}}\right) = S(r, g)$$

by (5.1), (5.2) and the claim($g = 0$ if and only if $g^{(k)} = a, g^{(k)} = g^{(k+1)} = 0$ whenever $g = 0$) we get a contradiction: $T(r, g) = S(r, g)$.

Hence the theorem. □

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